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On a Cluster Expansion for Lattice Spin Systems: A Finite-Size Condition for the Convergence

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A study is made of the statistical mechanics of classical lattice spin systems with finite-range interactions in two dimensions. By means of a decimation procedure, a finite-size condition is given for the convergence of a cluster expansion that is believed to be useful for treating the range of temperature between the critical one T_c and the estimated threshold T_0 of convergence of the usual high-temperature expansion.

KEY WORDS: Lattice spin systems: cluster expansion; analyticity; finite-size conditions.

1. INTRODUCTION

In this paper we study some aspects of the statistical mechanics of a class of lattice spin systems. To introduce the kind of problem we want to deal with, consider the case of a standard ferromagnetic spin system above its critical temperature $T_c = \inf\{T$: spontaneous magnetization $m^*(T) = 0\}$. The following features are expected to hold for these systems: uniqueness of the Gibbs state, rapid decay of correlations, and analyticity of thermodynamic functions; but all these nice properties are far from being proved in general. The theory of the high-temperature pure phase has been developed in general only for very weakly coupled systems.

The usual approaches, such as the Gallavotti–Miracle Sole[6] equations, are useful far from the critical point; in other words, they work only for temperatures $T \geqslant T_0$ where T_0 is strictly larger than T_c. The above high-temperature expansions are basically perturbation theories around a

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reference system composed of independent spins. In order to get convergence of the series expansions, all the interactions (even among nearest neighbor spins) are required to be small and so the basic length scale in that case is just the lattice spacing. Let us call "intermediate temperatures" the values T in the range $T_c < T < T_0$.

The aim of the present paper is to provide, in some cases, a perturbation theory that, at least in principle, can allow one to treat the intermediate temperature region. For technical reasons we limit ourselves to the two-dimensional case; moreover, for the sake of simplicity of the exposition, we only consider short-range spin systems with finite single-spin state. Generalizations including long-range, rapidly decaying interactions as well as general compact spin systems do not present any particular difficulty. On the other hand, the extension of our approach to three or more dimensions certainly seems to be possible, but it is somehow involved from a geometric point of view. We hope to be able in the future to simplify these geometric aspects. In this paper we just present some useful ideas for investigating the intermediate temperature region. We cannot really prove a general convergence result for a perturbative theory in this range of temperatures. For the moment we only give a "constructive criterion" similar to the one introduced by Dobrushin and Shlosman[4]: as in their approach, we reduce the problem of the convergence to an explicit condition that can be verified by means of a computer.

This condition refers to the statistical mechanical behavior of a finite-size system. One can reasonably expect that for any temperature $T > T_c$ there exists a size so large that our condition is satisfied for this size (see below). In this way any temperature $T > T_c$ could be treated via a convergent cluster expansion, but of course the size needed to verify the condition would have to increase to infinity as $T \to T_c$.

Let us now explain the basic ideas of our approach. The starting point of our analysis is the following, quite trivial, remark: a lattice system above its critical temperature T_c behaves like a weakly coupled system on a scale large with respect to the correlation length at this temperature. To show this behavior, we adopt a renormalization group philosophy. We apply to our system a "decimation procedure" (see, for instance, Ref. 9) and in this way we get a physically equivalent system whose effective interaction, in some particular circumstances, can be weaker than the original one. This effective decreasing of the interaction is expected to hold when the size of the blocks that are involved in the decimation transformation is of the same order of magnitude as the correlation length.

As is well known, the renormalization group methods were introduced in order to study the critical phenomena, but there are also applications in which a finite number of renormalization group transformations are used to analyze a given system in the proper *finite* length scale and in this way to reach the weak coupling region. A simple example is provided by the one-dimensional, short-range Ising systems. Even in this trivial case, where $T_c = 0$, the usual cluster expansions converge absolutely only for T sufficiently large; nevertheless, for any $T > 0$, after a suitable decimation procedure on a scale $L(T)$ (large in comparison with the inverse of the gap of the transfer matrix at the corresponding temperature), we get an effective interaction which is weak enough to allow the convergence of a cluster expansion. See Refs. 2 and 3 for more details (in those papers the more complicated case of one-dimensional systems with long-range interactions decaying like $1/r^{2+\varepsilon}$ was considered). Similar arguments are also used in percolation theory: see, for instance, Ref. 11.

Let us now come back to the present work. As we have said, we give an explicit condition for the convergence; it is related to some particular finite-volume mixing properties: a "small parameter" for the expansion will be provided by a sort of finite-volume truncated correlation function. Moreover, the reference system around which we perform our perturbative expansion is not universal (as in the standard high-temperature expansions): it is model-dependent and is related to some finite-volume system with nontrivial correlations.

The paper is organized as follows: in Section 2 we give the basic definitions, transform our original system into a polymer system by means of a block decimation procedure, and state the main result. In Section 3 we study some sufficient conditions for the convergence of the cluster expansions and discuss the equivalence with the conditions of Dobrushin and Shlosman.

2. DEFINITIONS AND RESULTS

Given $\Lambda \subset Z^2$, the configuration space in Λ is the set $S_\Lambda = \{0, 1,..., n\}^\Lambda$, $n \subset N$: a configuration in S_Λ is a map $\sigma_\Lambda : \Lambda \to \{0, 1,..., n\}$. We denote by $|\Lambda|$ the cardinality of a finite set $\Lambda \subset Z^2$. We suppose, given a potential $U = \{U_X, X \subset Z^2, |X| < \infty\}$, $U_X : S_X \to R$, such that:

(i) $\exists r_0 > 0$: $U_X = 0$ if diam $X > r_0$ (finite range).

(ii) $\forall X \subset Z^2$, $|X| < \infty$, $\forall k \in Z^2$: $U_{X+k} = U_X$ (translation invariance).

Given a finite volume $\Lambda \subset Z^2$, we denote by $H_\Lambda(\sigma_\Lambda)$ the energy associated to the generic configuration $\sigma_\Lambda \in S_\Lambda$ multiplied by $-1/T$ (T being the temperature). It is given by

$$H_\Lambda(\sigma_\Lambda) = -\frac{1}{T} \sum_{X \subset \Lambda} U_X(\sigma_X) \tag{2.1}$$

Given two disjoint finite regions Λ_1 and Λ_2, we define the interaction between Λ_1 and Λ_2 as the real function W_{Λ_1, Λ_2} on $S_{\Lambda_1} \otimes S_{\Lambda_2}$ given by

$$W_{\Lambda_1, \Lambda_2}(\sigma_{\Lambda_1}, \sigma_{\Lambda_2}) = H_{\Lambda_1 \cup \Lambda_2}(\sigma_{\Lambda_1}, \sigma_{\Lambda_2}) - H_{\Lambda_1}(\sigma_{\Lambda_1}) - H_{\Lambda_2}(\sigma_{\Lambda_2}) \tag{2.2}$$

Of course $W_{\Lambda_1, \Lambda_2} \equiv 0$ if $\text{dist}(\Lambda_1, \Lambda_2) > r_0$.

The finite-volume Gibbs measure with "empty boundary conditions" (no interaction with the exterior) is given by

$$\mu_\Lambda(\sigma_\Lambda) = \frac{\exp H_\Lambda(\sigma_\Lambda)}{Z_\Lambda} \tag{2.3a}$$

$$Z_\Lambda = \sum_{\sigma_\Lambda \in S_\Lambda} \exp H_\Lambda(\sigma_\Lambda) \tag{2.3b}$$

Given a finite $\Lambda \subset Z^2$, we call the "outer boundary" of Λ the set

$$\partial\Lambda = \{k \in Z^2 \backslash \Lambda : \text{dist}(k, \Lambda) \leqslant r_0\} \tag{2.4}$$

Given a spin configuration $\tau \in S_{\partial\Lambda}$ the finite-volume Gibbs measure in Λ with boundary condition τ is given by

$$\mu_\Lambda^\tau(\sigma_\Lambda) = \frac{\exp[H_\Lambda(\sigma_\Lambda) + W_{\Lambda, \partial\Lambda}(\sigma_\Lambda, \tau)]}{Z_\Lambda^\tau} \tag{2.5}$$

$$Z_\Lambda^\tau = \sum_{\sigma_\Lambda \in S_\Lambda} \exp[H_\Lambda(\sigma_\Lambda) + W_{\Lambda, \partial\Lambda}(\sigma_\Lambda, \tau)]$$

The quantity $Z_\Lambda(Z_\Lambda^\tau)$ is called the partition function in Λ with empty (τ) boundary conditions.

Let us now introduce a partition of Z^2 into blocks. Suppose L is an odd integer. For $k \equiv (k_1, k_2) \in Z^2$ ($k_1, k_2 \in Z$ being the coordinates of k) with $k_1 + k_2$ even, we consider the set

$$\bar{A}_k = \left\{h \equiv (h_1, h_2) \in Z^2 : -\frac{L-1}{2} + k_{1,2}L \leqslant h_{1,2} \leqslant \frac{L-1}{2} + k_{1,2}L\right\}$$

For $k \equiv (k_1, k_2) \in Z^2$ with $k_1 + k_2$ odd we consider the set

$$\bar{B}_k = \left\{h \equiv (h_1, h_2) \in Z^2 : -\frac{L-1}{2} + k_{1,2}L \leqslant h_{1,2} \leqslant \frac{L-1}{2} + k_{1,2}L\right\}$$

Suppose l is an even integer $< L$. For any $k \equiv (k_1, k_2) \in Z^2$ we define

$$\bar{C}_k = \left\{h \equiv (h_1, h_2) \in Z^2 : -\frac{l-1}{2} + \frac{L}{2} + k_{1,2}L \leqslant h_{1,2} \leqslant \frac{l-1}{2} + \frac{L}{2} + k_{1,2}L\right\}$$

If we associate to each lattice site $k \in Z^2$ the square of edge 1 and center k, then the sets $\bar{A}k$, $\bar{B}_{k'}$, $k \equiv (k_1, k_2)$, $k' \equiv (k'_1, k'_2)$, can be identified as the squares of edge L and centers $x \equiv (k_1 L, k_2 L)$, $x' \equiv (k'_1 L, k'_2 L)$, respectively, whereas C_k becomes the square of center $x \equiv (k_1 L + L/2, k_2 L + L/2)$ and edge l.

The set of all the $\bar{A}_k$ and $\bar{B}_k$ constitutes a chessboard partition of Z^2; any C_k overlaps two $\bar{A}$ blocks and two $\bar{B}$ blocks. We define the blocks

$$A_k = \bar{A}_k \Big\backslash \left[\bar{A}_k \cap \left(\bigcup_{h \in Z^2} C_h \right) \right]$$

$$B_k = \bar{B}_k \Big\backslash \left[\bar{B}_k \cap \left(\bigcup_{h \in Z^2} C_h \right) \right]$$

We have

$$Z^2 = \left(\bigcup_{k_1 + k_2 \text{ even}} A_k \right) \cup \left(\bigcup_{k'_1 + k'_2 \text{ odd}} B_{k'} \right) \cup \left(\bigcup_{k'' \in Z^2} C_{k''} \right) \tag{2.6}$$

We will choose L, l

$$L/2 < l \leqslant L - 1 \tag{2.7}$$

in such a way that the interaction between any two blocks of the same kind vanishes: $L/2 > r_0$ is a possible choice. Given a set $\Omega \subseteq Z^2$, we define

$$\begin{aligned} a(\Omega) &= \{ A_k \mid A_k \subset \Omega \} \\ b(\Omega) &= \{ B_k \mid B_k \subset \Omega \} \\ c(\Omega) &= \{ C_k \mid C_k \subset \Omega \} \\ a'(\Omega) &= \{ A_k \mid \operatorname{dist}(A_k, \Omega) \leqslant 1 \} \\ b'(\Omega) &= \{ B_k \mid \operatorname{dist}(B_k, \Omega) \leqslant 1 \} \\ c'(\Omega) &= \{ C_k \mid \operatorname{dist}(C_k, \Omega) \leqslant 1 \} \end{aligned} \tag{2.8}$$

Given a set of blocks Γ (of A, B, or C type), we define the support $\tilde{\Gamma}$ of Γ as the subset of Z^2 given by the union of the blocks of Γ. So we have

$$\tilde{a}(\Omega) = \bigcup_{A_k \in a(\Omega)} A_k = \bigcup_{A_k \subset \Omega} A_k$$

Analogous definitions hold for $\tilde{b}(\Omega)$, $\tilde{c}(\Omega)$, $\tilde{a}'(\Omega)$, $\tilde{b}'(\Omega)$, and $\tilde{c}'(\Omega)$.

Notation. α_k, β_k, γ_k will denote the spin configuration in A_k, B_k, C_k, respectively ($\alpha_k \in S_{A_k}$, etc.). Given a set $\Omega \subseteq Z^2$, α_Ω, β_Ω, γ_Ω will denote spin configurations in $\tilde{a}(\Omega)$, $\tilde{b}(\Omega)$, $\tilde{c}(\Omega)$; α, β, γ denote spin configurations in the union of all A, B, C blocks, respectively:

$$\alpha \in S_{\tilde{a}(Z^2)} \quad \text{etc.}$$

We often use α, β, γ to denote arguments of cylindrical functions that in fact depend only on the restriction of configurations to particular subsets that will be understood from the context.

We set

$$\begin{aligned} H(\alpha) &\equiv H_{A_k}(\alpha_k) \\ H(\beta_k) &\equiv H_{B_k}(\beta_k) \\ H(\gamma_k) &\equiv H_{C_k}(\gamma_k) \end{aligned} \tag{2.9}$$

Given $\Omega \subseteq Z^2$, we set

$$\begin{aligned} H_\Omega(\alpha) &= \sum_{A_k \in a(\Omega)} H(\alpha_k) \\ H_\Omega(\beta) &= \sum_{B_k \in b(\Omega)} H(\beta_k) \\ H_\Omega(\gamma) &= \sum_{C_k \in c(\Omega)} H(\gamma_k) \end{aligned} \tag{2.10}$$

Moreover, we denote by $W(\gamma_k, \beta)$ the interaction between the block C_k and the contiguous B blocks and by $W(\gamma_k, \alpha_\beta)$ the interaction between the block C_k and the contiguous A and B blocks:

$$\begin{aligned} W(\gamma_k, \beta) &= W_{C_k, \tilde{b}'(C_k)}(\gamma_k, \beta) \\ W(\gamma_k, \alpha\beta) &= W_{C_k, \tilde{b}'(C_k) \cup \tilde{a}'(C_k)}(\gamma_k, (\beta\alpha)) \end{aligned} \tag{2.11}$$

We set

$$W_\Omega(\gamma, \beta) = \sum_{C_k \in c(\Omega)} W(\gamma_k, \beta) \tag{2.12}$$

Analogous definitions hold for $W(\beta_k, \alpha)$, $W_\Omega(\beta, \alpha)$, $W_\Omega(\gamma, \alpha\beta)$, etc.

Notice that generally $W_\Omega(\beta, \alpha) \neq W_\Omega(\alpha, \beta)$.

To simplify the notation, we often write $\sum_\alpha$ for $\sum_{\alpha \in S_{\tilde{a}(\Lambda)}}$; $\sum_{\alpha_k}$ for $\sum_{\alpha_k \in S_{A_k}}$; etc.

We consider a system enclosed in a box $\Lambda \equiv \Lambda_p$ defined in the following way: let Q_p be the square given by

$$Q_p = \left\{ \left(h \equiv (h_1, h_2) \in Z^2 \colon -\frac{pl-1}{2} L \leqslant h_{1,2} \leqslant +\frac{pl-1}{2} \right) \right\}$$

Then

$$\Lambda_p = \tilde{a}(Q_p) \cup \tilde{b}(Q_p) \cup \tilde{c}(Q_p)$$

We notice that the shape of Λ_p is approximately but not exactly that of a square; in fact, Λ_p is obtained from Q_p by eliminating the C blocks that intersect the boundary and are not completely contained in Q_p.

We consider empty boundary conditions and we want to compute the corresponding partition function in Λ_p [see Eq. (2.3b)]. By integrating first over the γ variables, then over the β's, and eventually over the α's, we transform our original spin system into a polymer system (see Refs. 8 and 10). Notice that L and l are at the moment free parameters (with the restrictions $L/2 \leqslant l < L-1$, $L/2 > r_0$). We can write

$$\begin{aligned} Z_\Lambda &= \sum_{\sigma_\Lambda \in S_\Lambda} \exp[H_\Lambda(\sigma_\Lambda)] \\ &= \sum_\alpha \exp[H_\Lambda(\alpha)] \sum_\beta \exp[H_\Lambda(\beta) + W_\Lambda(\beta, \alpha)] \\ &\quad \times \sum_\gamma \exp[H_\Lambda(\gamma) + W_\Lambda(\gamma, \alpha\beta)] \end{aligned} \tag{2.13}$$

[see Eqs. (2.9)–(2.12)].

By performing the sum over the γ variables, we get

$$Z_\Lambda = \sum_\alpha \exp[H_\Lambda(\alpha)] \sum_\beta \exp(H_\Lambda(\beta) + W_\Lambda(\beta, \alpha)] \prod_{C_k \in c(\Lambda)} Z_{C_k}(\beta, \alpha) \tag{2.14}$$

where

$$Z_{C_k}(\beta, \alpha) \equiv Z_{C_k}^{\beta, \alpha} = \sum_{\gamma_k} \exp[H(\gamma_k) + W(\gamma_k, \alpha\beta)] \tag{2.15}$$

Notation. Given any finite set $\Omega \subset Z^2$ we order the A blocks in $a(\Omega)$ according to the lexicographic order of their centers: $(k_1, k_2) < (k_1', k_2')$ if $k_1 < k_1'$ or $k_1 = k_1'$, $k_2 < k_2'$, and we write

$$a(\Omega) = A_1(\Omega),\ A_2(\Omega), \ldots,\ A_N(\Omega)$$

where $N = |a(\Omega)|$ is the number of A blocks in $a(\Omega)$. We do the same for $b(\Omega)$, $c(\Omega)$, $a'(\Omega)$, $b'(\Omega)$, $c'(\Omega)$.

So, given a block C_k, we have

$$b(C_k) = B_1(C_k),\ B_2(C_k); \qquad a(C_k) = A_1(C_k),\ A_2(C_k)$$

For $\beta_1 \in S_{B_1}(C_k)$, $\beta_2 \in S_{B_2}(C_k)$, $\alpha_1 \in S_{A_1}(C_k)$, $\alpha_2 \in S_{A_2}(C_k)$, we write explicitly

$$Z_{C_k}(\beta, \alpha) = Z_{C_k}(\beta_1, \beta_2, \alpha_1, \alpha_2)$$

We use the trivial identity

$$\begin{aligned} Z_{C_k}(\beta_1, \beta_2, \alpha_1, \alpha_2) &= [\psi_{C_k}(\beta, \alpha) + 1] \\ &\times \frac{Z_{C_k}(\beta_1, 0, \alpha_1, \alpha_2)\, Z_{C_k}(0, \beta_2, \alpha_1, \alpha_2)\, Z_{C_k}(0, 0, 0, 0)}{Z_{C_k}(0, 0, \alpha_1, 0)\, Z_{C_k}(0, 0, 0, \alpha_2)} \end{aligned} \tag{2.16}$$

where

$$\begin{aligned} \psi_{C_k}(\beta, \alpha) &= \psi_{C_k}(\beta_1, \beta_2, \alpha_1, \alpha_2) \\ &= \frac{Z_{C_k}(\beta_1, \beta_2, \alpha_1, \alpha_2)\, Z_{C_k}(0, 0, \alpha_1, \alpha_2)}{Z_{C_k}(\beta_1, 0, \alpha_1, \alpha_2)\, Z_{C_k}(0, \beta_2, \alpha_1, \alpha_2)} \\ &\quad \times \frac{Z_{C_k}(0, 0, \alpha_1, 0)\, Z_{C_k}(0, 0, 0, \alpha_2)}{Z_{C_k}(0, 0, \alpha_1, \alpha_2)\, Z_{C_k}(0, 0, 0, 0)} - 1 \end{aligned} \tag{2.17}$$

and the symbol 0 in the argument of Z_{C_k} means that in the corresponding block we have chosen the configuration $\sigma_x = 0$.

For example, in the case $Z_{C_k}(\beta_1, 0, \alpha_1, \alpha_2)$ we have the partition function in the block C_k with boundary conditions given by β_1 in $B_1(C_k)$, 0 in $B_2(C_k)$, α_1 in $A_1(C_k)$, α_2 in $A_2(C_k)$.

Notice that here 0 only plays the role of an arbitrarily fixed reference configuration. By Eqs. (2.14)–(2.17) we get

$$\begin{aligned} Z_\Lambda = & \prod_{C_k \in c(\Lambda)} Z_{C_k}(0) \sum_\alpha \exp[H_\Lambda(\alpha)] \\ & \times \prod_{A_k \in a(\Lambda)} [\bar{Z}_k(\Lambda, \alpha)]^{-1} \sum_\beta \exp[H_\Lambda(\beta) + W_\Lambda(\beta, \alpha)] \\ & \times \prod_{B_k \in b(\Lambda)} \tilde{Z}_k(\Lambda, \beta_k, \alpha) \prod_{C_k \in c(\Lambda)} [\psi_{C_k}(\beta, \alpha) + 1] \end{aligned} \tag{2.18}$$

where $\bar{Z}_k(\Lambda, \alpha_k)$ and $\tilde{Z}_k(\Lambda, \beta_k, \alpha)$ are defined in the following way: If A_k is in the bulk, namely $\mathrm{dist}(A_k, \Lambda^c) > 1$ and $C_1(A_k)$, $C_2(A_k)$, $C_3(A_k)$, $C_4(A_k)$ are the blocks in $c'(A_k)$ (in lexicographic order), then

$$\begin{aligned} \bar{Z}_k(\Lambda, \alpha_k) \equiv \bar{Z}_k(\alpha_k) &= Z_{C_1}(0, 0, 0, \alpha_k)\, Z_{C_2}(0, 0, 0, \alpha_k) \\ &\quad \times Z_{C_3}(0, 0, \alpha_k, 0)\, Z_{C_4}(0, 0, \alpha_k, 0) \end{aligned} \tag{2.19}$$

If A_k is adjacent to the boundary $\partial\Lambda$, $\bar{Z}_k(\Lambda, \alpha_k)$ is defined with the obvious modifications. For instance, for

$$K = \left(\frac{p-1}{2}, \frac{p-1}{2}\right)$$

namely for the A block in the upright corner of Λ_p, we have

$$\bar{Z}_k(\Lambda, \alpha_k) = Z_{C_1}(0, 0, 0, \alpha_k)$$

Moreover, if B_k is in the bulk, $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the configurations in $A_1(B_k)$, $A_2(B_k)$, $A_3(B_k)$, $A_4(B_k)$ that form the set $a'(B_k)$ and $c'(B_k) = C_1(B_k)$, $C_2(B_k)$, $C_3(B_k)$, $C_4(B_k)$, then

$$\begin{aligned}\tilde{Z}_k(\Lambda, \beta_k, \alpha) &\equiv \tilde{Z}(\beta_k, \alpha) \\ &= Z_{C_1}(0, \beta_k, \alpha_1, \alpha_2)\, Z_{C_2}(0, \beta_k, \alpha_1, \alpha_3) \\ &\quad \times Z_{C_3}(\beta_k, 0, \alpha_2, \alpha_4)\, Z_{C_4}(\beta_k, 0, \alpha_3, \alpha_4) \end{aligned} \tag{2.20}$$

Again for B_k adjacent to the boundary the expression of $\tilde{Z}$ is modified in an obvious way.

Now, for any given configuration $\alpha \in S_{\bar{a}(Z^2)}$ and for any $\Omega \subset b(\Lambda)$, consider the normalized measure $\mu_{\Lambda,\Omega,\alpha}$ on $S_{\bar{\Omega}}$ given by

$$\mu_{\Lambda,\Omega,\alpha}(\beta) = \frac{\exp[H_\Omega(\beta) + W_\Omega(\beta, \alpha)] \prod_{B_k \in \Omega} \tilde{Z}_k(\Lambda, \beta_k, \alpha)}{\sum_\beta \exp[H_\Omega(\beta) + W_\Omega(\beta, \alpha)] \prod_{B_k \in \Omega} \tilde{Z}_k(\Lambda, \beta_k, \alpha)} \tag{2.21}$$

[see Eq. (2.10)].

We notice that μ is a product measure:

$$\mu_{\Lambda,\Omega,\alpha}(\beta) = \prod_{B_k \in \Omega} \mu_{\Lambda,\beta_k,\alpha}(\beta_k) \tag{2.22}$$

with

$$\mu_{\Lambda,\beta_k,\alpha}(\beta) = \frac{\exp[H(\beta_k) + W(\beta_k, \alpha)]\, \tilde{Z}_k(\Lambda, \beta_k, \alpha)}{\hat{Z}_{B_k}(\Lambda, \alpha)} \tag{2.23}$$

where

$$\begin{aligned}\hat{Z}_{B_k}(\Lambda, \alpha) &= \sum_{\beta_k} \exp[H(\beta_k) + W(\beta_k, \alpha)]\, \tilde{Z}_k(\Lambda, \beta_k, \alpha) \\ &= \sum_{\gamma\beta_k} \exp\Bigg[H(\beta_k) + W(\beta_k, \alpha) \\ &\quad + \sum_{C_h \in c'(B_k) \cap c(\Lambda)} H(\gamma_h) + W(\gamma_h, \alpha, \beta')\Bigg]\end{aligned} \tag{2.24}$$

and

$$\beta'|_{B_k} = \beta_k, \qquad \beta'|_{\bar{b}(c'(B_k)\setminus B_k)} = 0$$

We write for B_k in the bulk

$$\hat{Z}_{B_k}(\Lambda, \alpha) \equiv \hat{Z}_{B_k}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \tag{2.25}$$

if $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are, respectively, the four spin configurations in the four A blocks of $a'(B_k)$. Again if B_k is adjacent to $\partial\Lambda$, we introduce the obvious changes. From Eqs. (2.18), (2.21)–(2.23), we get

$$\begin{aligned} Z_\Lambda = &\prod_{C_k \in c(\Lambda)} Z_{C_k}(0) \sum_\alpha \exp[H_\Lambda(\alpha)] \prod_{A_k \in a(\Lambda)} [\bar{Z}_k(\Lambda, \alpha_k)^{-1}] \\ &\times \prod_{B_k \in b(\Lambda)} \hat{Z}_{B_k}(\Lambda, \alpha) \sum_{\Gamma \subset c(\Lambda)} \sum_\beta \mu_{\Lambda, b'(\Gamma), \alpha}(\beta) \prod_{C_k \in \Gamma} \Psi_{C_k}(\beta, \alpha) \end{aligned} \tag{2.26}$$

where in the sum over Γ we include the case $\Gamma = \varnothing$ and we set $\prod_{C_k \in \varnothing} = 1$.

Now, for B_k in the bulk, we use the trivial identity

$$\begin{aligned} &\hat{Z}_{B_k}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \\ &\quad = [\Phi_{B_k}(\alpha) + 1] \\ &\quad\quad \times \hat{Z}_{B_k}(\alpha_1, 0, 0, 0)\, \hat{Z}_{B_k}(0, \alpha_2, 0, 0) \\ &\quad\quad \times \hat{Z}_{B_k}(0, 0, \alpha_3, 0)\, \hat{Z}_{B_k}(0, 0, 0, \alpha_4) \\ &\quad\quad \times [\hat{Z}_{B_k}(0, 0, 0, 0)]^{-3} \end{aligned} \tag{2.27}$$

where

$$\begin{aligned} \Phi_{B_k}(\alpha) &= \Phi_{B_k}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \\ &= \hat{Z}_{B_k}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)[\hat{Z}_{B_k}(0, 0, 0, 0)]^3 \\ &\quad \times [\hat{Z}_{B_k}(\alpha_1, 0, 0, 0)\, \hat{Z}_{B_k}(0, \alpha_2, 0, 0) \\ &\quad \times \hat{Z}_{B_k}(0, 0, \alpha_3, 0)\, \hat{Z}_{B_k}(0, 0, 0, \alpha_4)]^{-1} - 1 \end{aligned} \tag{2.28}$$

If B_k is adjacent to the boundary, we use similar identities. Now, if A_k is in the bulk and $B_1(A_k)$, $B_2(A_k)$, $B_3(A_k)$, $B_4(A_k)$ are the four B blocks in $b'(A_k)$, we set

$$\begin{aligned} Z^*(\alpha_k) = &\hat{Z}_{B_1}(0, 0, 0, \alpha_k)\, \hat{Z}_{B_2}(0, 0, \alpha_k, 0) \\ &\times \hat{Z}_{B_3}(0, \alpha_k, 0, 0)\, \hat{Z}_{B_4}(\alpha_k, 0, 0, 0) \end{aligned} \tag{2.29}$$

In general (for A_k not necessarily in the bulk) we call $Z_k^*(\Lambda, \alpha_k)$, $\Phi_{B_k}(\Lambda, \alpha)$ the analogous quanities $[Z_k^*(\Lambda, \alpha_k) \equiv Z^*(\alpha_k)$, $\Phi_{B_k}(\Lambda, \alpha) = \Phi_{B_k}(\alpha)$ for A_k in the bulk].

From Eqs. (2.26)–(2.29) we get

$$\begin{aligned} Z_\Lambda = & \prod_{C_k \in c(\Lambda)} Z_{C_k}(0) \prod_{B_k \in b(\Lambda)} [\bar{Z}_{B_k}(\Lambda, 0)]^{-3} \\ & \times \sum_{\alpha} \prod_{A_k \in a(\Lambda)} \exp[H(\alpha_k)] [\bar{Z}_k(\Lambda, \alpha_k)^{-1} Z_k^*(\Lambda, \alpha_k)] \\ & \times \sum_{\Delta \subset b(\Lambda)} \prod_{B_k \in \Delta} \Phi_{B_k}(\Lambda, \alpha) \\ & \times \sum_{\Gamma \subset c(\Lambda)} \sum_{\beta} \mu_{\Lambda, b'(\Gamma), \alpha}(\beta) \prod_{C_k \in \Gamma} \Psi_{C_k}(\beta, \alpha) \end{aligned} \tag{2.30}$$

where again the sum over Δ includes $\Delta = \varnothing$ and $\prod_{\Delta \in \varnothing} = 1$.

Now we define the probability measure $\nu_{k,\Lambda}$ on S_{A_k} as

$$\nu_{k,\Lambda}(\alpha_k) = \frac{\exp[H(\alpha_k)] [\bar{Z}_k(\Lambda, \alpha_k)]^{-1} Z_k^*(\Lambda, \alpha_k)}{\sum_{\alpha_k'} \exp[H(\alpha_k')] [\bar{Z}_k(\Lambda, \alpha_k')]^{-1} Z_k^*(\Lambda, \alpha_k')} \tag{2.31}$$

We set

$$\rho_{k,\Lambda} = \sum_{\alpha_k} \exp[H(\alpha_k)] [\bar{Z}_k(\Lambda, \alpha_k)]^{-1} Z_k^*(\Lambda, \alpha_k) \tag{2.32}$$

For any $\Omega \subseteq a(\Lambda)$ we introduce the product measure on $S_{\bar{\Omega}}$ given by

$$\nu_{\Lambda,\Omega}(\alpha) = \prod_{A_k \in \Omega} \nu_{k,\Lambda}(\alpha_k) \tag{2.33}$$

By using Definitions 2.3.1–2.3.3, we can write

$$\begin{aligned} Z_\Lambda = & \prod_{C_k \in c(\Lambda)} Z_{C_k}(0) \prod_{B_k \in b(\Lambda)} [\hat{Z}_{B_k}(\Lambda, 0)]^{-3} \\ & \times \prod_{A_k \in a(\Lambda)} \rho_{k,\Lambda} \sum_{\Delta \subset b(\Lambda)} \sum_{\Gamma \subset c(\Lambda)} \sum_{\alpha} \nu_{\Lambda, a'(\bar{\Delta}) \cup a'(\bar{\Gamma})}(\alpha) \\ & \times \prod_{B_k \in \Delta} \Phi_{B_k}(\Lambda, \alpha) \sum_{\beta} \mu_{\Lambda, b'(\bar{\Gamma}), \alpha^{(\beta)}} \prod_{C_k \in \Gamma} \Psi_{C_k}(\beta, \alpha) \end{aligned} \tag{2.34}$$

Now we are almost ready to write down our partition function in terms of a gas of polymers whose only interaction is a hard core exclusion. We see from Eqs. (2.17), (2.28) that a term Ψ_{C_k} corresponds to a four-body interaction among the A and B blocks adjacent to C_k and that the term

Φ_{B_k} corresponds, for B_k in the bulk, to a four-body interaction among the four A blocks adjacent to B_k. Moreover, looking at Eq. (2.21) we see that the measure $\mu_{A,b'(\tilde{\Gamma}),\alpha}$ depends on the spin configurations in all the A blocks adjacent to $\tilde{b}'(\tilde{\Gamma})$, namely in $\tilde{a}(\tilde{b}'(\tilde{\Gamma}))$; then, so to say, a Ψ_{C_k} term "extends its action to the whole set $\tilde{a}'(\tilde{b}'(C_k))$. So we define two kind of bonds:

1. A C_k bond is the set of A and B blocks given by $a'(C_k)$, $b'(C_k)$, $a'(\tilde{b}'(C_k)) \cap a(\Lambda)$.
2. A B_k bond is the set of A blocks $a(B_k) \cap a(\Lambda)$.

The support $\tilde{l}$ of a bond l is the subset of Z^2 obtained as the union of the blocks belonging to l [see definition after Eq. (2.8)].

Definition 2.1. Two bonds l_1, l_2 are said to be connected if $\tilde{l}_1 \cap \tilde{l}_2 \neq \varnothing$.

Definition 2.2. A *polymer* R is a set of bonds $l_1,..., l_k$ that is connected in the sense that $\forall ij$: $1 \leqslant i < j \leqslant k$ there exists a chain of connected bonds in R joining l_i to l_j. The support $\tilde{R}$ of a polymer $R = l_1,..., l_k$ is simply $\tilde{R} = \bigcup_{i=1}^{k} \tilde{l}_i$: We call $\mathscr{R}$ the set of all the possible polymers with arbitrary support in Z^2 and $\mathscr{R}_\Lambda$ the set of all the polymers such that $\tilde{R} \subset \Lambda$.

Definition 2.3. Two polymers R_i, R_j are said to be compatible if $\tilde{R}_i \cap \tilde{R}_j = \varnothing$. Otherwise they are called incompatible. We denote a C_K bond or a B_k bond simply by C_k, B_k (with abuse of notation).

Definition 2.4. Given a polymer $R = C_{k_1},..., C_{k_r}, B_{h_1},..., B_{h_s}$, $R \in \mathscr{R}_\Lambda$, we call "finite-volume activity of R in Λ" the quantity

$$\zeta_\Lambda = \sum_{\alpha} \nu_{\Lambda, a(\tilde{R})}(\alpha) \prod_{B_k \in R} \Phi_{B_k}(\Lambda, \alpha) \sum_{\beta} \mu_{\Lambda, b(\tilde{R}), \alpha}(\beta) \prod_{C_k \in R} \Psi_{C_k}(\beta, \alpha) \quad (2.35)$$

For a generic polymer $R = C_{k_1},..., C_{k_r}, B_{h_1},..., B_{h_s}$, $R \in \mathscr{R}$, we call simply the "activity of R" the quantity

$$\zeta(R) = \sum_{\alpha} \nu(\alpha) \prod_{B_k \in R} \Phi_{B_k}(\alpha) \sum_{\alpha} \mu_\alpha(\beta) \prod_{C_k \in R} \Psi_{C_k}(\beta, \alpha) \quad (2.36)$$

where:

(i) ν is the product measure on $S_{\tilde{a}(Z^2)}$,

$$\nu = \bigotimes_{A_k \in a(Z^2)} \nu_k$$

with

$$\nu_k(\alpha_k) = \frac{\exp[H(\alpha_k)] [\bar{Z}(\alpha_k)]^{-1} Z^*(\alpha_k)}{\sum_{\alpha'_k} \exp[H(\alpha'_k)] [\bar{Z}(\alpha'_k)]^{-1} Z^*(\alpha'_k)} \quad (2.37)$$

where $\bar{Z}$, Z^* are just the expression valid in the bulk given by Eqs. (2.19), (2.29).

(ii) μ is a function on $S_{\tilde{a}(Z^2)}$ with values the product measures on $S_{\tilde{b}(Z^2)}$ given by $\mu_\alpha = \bigotimes_k \mu_{k,\alpha}$ with

$$\mu_{k,\alpha}(\beta_k) = \frac{\exp[H(\beta_k) + W(\beta_k, \alpha)]\, \tilde{Z}(\beta_k \alpha)}{\sum_{\beta'_k} \exp[H(\beta'_k) + W(\beta'_k, \alpha)]\, \tilde{Z}(\beta'_k, \alpha)} \tag{2.38}$$

and $\tilde{Z}$ is the expression valid in the bulk given by Eq. (2.20).

Of course, if $\tilde{R}$ is contained in the bulk of Λ, then $\zeta_\Lambda(R) \equiv \zeta(R)$.

By using Definitions 2.2 and 2.4, from Eq. (2.34), we get

$$\begin{aligned} Z_\Lambda &\Bigg[\prod_{C_k \in c(\Lambda)} Z_{C_k}(0) \prod_{B_k \in b(\Lambda)} \hat{Z}_{B_k}(\Lambda, 0)^{-3} \prod_{A_k \in a(\Lambda)} \rho_{\Lambda,k} \Bigg]^{-1} \\ &= 1 + \sum_{n \geqslant 1} \sum_{\substack{R_1,\ldots,R_n \in \mathscr{R}_\Lambda \\ \tilde{R}_i \cap \tilde{R}_j = \varnothing}} \prod_{i=1}^{n} \zeta_\Lambda(R_i) \end{aligned} \tag{2.39}$$

We see from Eq. (2.39) that the (suitably normalized) partition function is equal to the partition function of a gas of polymers with activity $\zeta_\Lambda(R)$ whose only interaction, given by the compatibility condition, is a hard core exclusion. The normalization factor

$$\mathscr{N} = \prod_{C_k \in c(\Lambda)} Z_{C_k}(0) \prod_{B_k \in b(\Lambda)} \hat{Z}_{B_k}(\Lambda, 0)^{-3} \prod_{A_k \in a(\Lambda)} \rho_{\Lambda,k}$$

describes our "reference system."

From Eqs. (2.3.5), (2.3.6) we immediately derive an estimate for the activity. For, suppose that there exists $\lambda > 0$ such that

$$\sup_k \sup_\alpha |\Phi_{B_k}(\Lambda, \alpha)| \leqslant \lambda, \qquad \sup_{\alpha, \beta} |\psi_{C_k}(\beta, \alpha)| \leqslant \lambda \tag{2.40}$$

then, if we denote by $|R|$ the number of bonds in R, namely for $R = C_{k_1},\ldots, C_{k_r}, B_{h_1},\ldots, B_{h_s}$: $|R| = r + s$, we get

$$|\zeta(R)| < \lambda^{|R|}, \qquad |\zeta_\Lambda(R)| < \lambda^{|R|} \tag{2.41}$$

The following theorem is true:

Theorem 2.5. Let

$$\Xi_\Lambda = 1 + \sum_{n \geqslant 1} \sum_{\substack{R_1,\ldots,R_n \in \mathscr{R}_\Lambda \\ \tilde{R}_i \cap \tilde{R}_j = \varnothing}} \prod_{i=1}^{n} \zeta_\Lambda(R_i)$$

Let

$$\varphi_T(R_1,\ldots, R_n)=\frac{1}{n!}\sum_{g\in G(R_1\cdots R_n)}(-1)^{\#\text{ edges in } g}$$

where $G(R_1,\ldots, R_n)$ is the set of connected graphs with n vertices $(1,\ldots, n)$ and edges ij corresponding to pairs R_iR_j such that $\tilde{R}_i\cap\tilde{R}_j\neq\emptyset$ (we set the sum equal to zero if G is empty and one if $n=1$). If

$$\lambda<\frac{1}{43}\frac{x}{1+x}e^{-x}\bigg|_{x=(5^{1/2}-1)/2} \tag{2.42}$$

then:

(i) There exists a positive constant $C(\lambda)$,

$$\sum_{\substack{R_1,\ldots, R_n\in\mathscr{R}\\ \exists R_i=R}}\varphi_T(R_1,\ldots, R_n)\prod_{i=1}^{n}\zeta(R_i)\leqslant C(\lambda)\left(\lambda\exp\frac{5^{1/2}-1}{2}\right)^{|R|} \tag{2.43}$$

(ii)
$$\Xi_\Lambda=\exp\left[\sum_{n\geqslant 1}\sum_{\substack{R_1,\ldots, R_n\\ \tilde{R}_i\subset\Lambda}}\varphi_T(R_1,\ldots, R_n)\prod_{i=1}^{n}\zeta_\Lambda(R_i)\right] \tag{2.44}$$

Proof. The proof can be obtained by the standard methods of the theory of the cluster expansion. We report here in a very short form the basic steps of the proof. We rely on the methods and results of Ref. 7.

Consider the set Γ of finite configurations $X=R_1,\ldots, R_n$ of polymers in $\mathscr{R}$ that are allowed to be incompatible and even to coincide.

A finite configuration is a function $X:\mathscr{R}\to N\cup\{0\}$ such that $N(X)=\sum_{R\in\mathscr{R}}X(R)<\infty$; $X(R)$ is called "multiplicity" of R in X. We denote by $\emptyset$ the empty configuration given by $X(R)=0\ \forall R$.

The sum X_1+X_2 of two configurations is simply given by

$$(X_1+X_2)(R)=X_1(R)+X_2(R)$$

Let F be the space of real functions f on Γ such that

$$\sup_{N(x)=n}|f(x)|<\infty,\qquad n=0, 1,\ldots$$

The convolution product of $f_1, f_2\in F$ is defined as

$$(f_1*f_2)(x)=\sum_{x_1+x_2=x}f_1(X_1)\,f_2(X_2)$$

Given $f\in F$ with $f(\emptyset)\neq 0$, f^{-1} is defined by

$$f^{-1}*f=\mathbf{1}$$

where

$$\mathbf{1}(x)=\begin{cases}1 & \text{if } x=\varnothing\\ 0 & \text{otherwise}\end{cases}$$

Let

$$\Phi(x)=\Phi(R_1,\dots, R_n)=\prod_{i=1}^{n}\zeta(R_i)\prod_{i<j}\chi(R_i, R_j)$$

where

$$\chi(R, R')=\begin{cases}1 & \text{if } \quad R, R' \text{ compatible}\\ 0 & \text{otherwise}\end{cases}$$

Let

$$\Delta_x(Y)=\sum_{Y_1+Y_2=Y}\Phi^{-1}(Y_1)\,\Phi(X+Y_2) \tag{2.45}$$

Let

$$I_m=\sup_{\substack{R_1,\dots,R_n\\ m\geqslant n\geqslant 1}}\sum_{\substack{Y\\ N(Y)=m-n}}|\Delta_{R_1,\dots,R_n}(Y)|\,(\lambda e^{-x})^{\sum_{i=1}^{n}|R_i|} \tag{2.46}$$

where x is a positive constant to be fixed later.

It is easy to see that the number of polymers containing a given A or B block such that $|R|=l$ is bounded by 43^l. Now, by using exactly the same arguments that lead to the estimates given by Eqs. (4.27) and (4.32) of Ref. 7, we see that if, for some $x>0$,

$$\frac{43\lambda e^x}{1-43\lambda e^x}<x$$

then the statement (i) is true with x in place of $(5^{1/2}-1)/2$. It is easy to see that to optimize the choice of x, we have to find the positive value of x that maximize the function

$$y=\frac{x}{1+x}e^{-x} \tag{2.47}$$

We get $x=(5^{1/2}-1)/2$. Of course, since we only control the activity via the estimate (2.41), the statement (i) holds with ζ_Λ in place of ζ. Statement (ii) follows easily from (i) (see Ref. 7 for more details). We notice that we can use the cluster expansion theory of Kotecky and Preiss and obtain a similar result with exactly the same condition (2.42) on λ (see Ref. 10, p. 493). ∎

We treated explicitly the partition function with zero boundary conditions, but it is clear that general boundary conditions as well as thermal averages of local observables can be treated by similar methods (see, for instance, Ref. 1). In particular, one can obtain a series expansion for any quantity of interest, such as the infinite-volume free energy or the correlation functions; exponential decay of truncated correlation functions and analyticity of thermodynamic functions and correlation functions can also be deduced by standard methods.

3. SUFFICIENT CONDITIONS FOR THE CONVERGENCE

In this section we give sufficient conditions for the convergence of our cluster expension. From Eqs. (2.17), (2.28), (2.39), (2.41), (2.42), we see that, for the convergence, we need some factorization properties of partition functions in sitable regions. The result of the following proposition says that it is possible to deduce these properties from a very simple factorization condition that involves only two conditioning spins.

Proposition 3.1. Let Ω be a finite, connected subset of Z, $\partial\Omega$ its outer boundary [see Eq. (2.4)], and τ a given spin configuration in $\partial\Omega$ ($\tau \in S_{\partial\Omega}$).

Let $D_1,..., D_l$, $l \geqslant 2$, be disjoint subsets of $\partial\Omega$ and $\delta_1,..., \delta_l$ some fixed spin configurations in $D_1,..., D_l$, respectively.

Let $Z^{\Omega}_{D_1 \cdots D_l}(\delta_1,..., \delta_l, \tau)$ be the partition function in Ω with boundary conditions given by (the restriction of) τ in $\partial\Omega \backslash \bigcup_{j=1}^{l} D_j$, and by δ_j in D_j, $j = 1,..., l$. (Namely, we substitute $\tau|_{D_j}$ with δ_j.)

Suppose that the following condition is satisfied:

Condition A. There exists a decreasing function $\mu: R^+ \to R^+$ with $\lim_{x \to \infty} \mu(x) = 0$ such that, for any finite Ω and for any pair of sites k, k' in $\partial\Omega$, we have

$$\sup_{\tau, \sigma_k, \sigma_{k'}} \left| \frac{Z^{\Omega}_{k,k'}(\sigma_k, \sigma_{k'}, \tau)\, Z^{\Omega}(\tau)}{Z^{\Omega}_{k}(\sigma_k, \tau)\, Z^{\Omega}_{k'}(\sigma_{k'}, \tau)} - 1 \right| < \mu(\text{dist}(k, k')) \tag{3.1}$$

with $Z^{\Omega}(\tau)$ simply the partition function in Ω with boundary conditions τ in $\partial\Omega$; then, $\forall\tau$, $\forall D_1,..., D_l$, $\delta_1,..., \delta_l$:

$$[1 - \mu(\rho)]^{\gamma(D_1,..., D_l)} \leqslant \frac{Z^{\Omega}_{D_1,..., D_l}(\delta_1,..., \delta_l, \tau)[Z^{\Omega}(\tau)]^{l-1}}{\prod_{j=1}^{l} Z^{\Omega}_{D_j}(\delta_j, \tau)} \leqslant [1 + \mu(\rho)]^{\gamma(D_1,..., D_l)} \tag{3.2}$$

where

$$\rho = \min_{1 \leqslant i < j \leqslant l} \operatorname{dist}(D_i, D_j) \tag{3.3}$$

$$\gamma(D_1,..., D_l) = \sum_{1 \leqslant i < j \leqslant l} |D_i| \, |D_j| \tag{3.4}$$

Proof. The proof is obtained by induction on the number $N = \sum_i |D_i|$. We distinguish two cases:

(i) We suppose that the estimates (3.2) are true for $D_1,..., D_l$, $\delta_1,..., \delta_l$ and we want to prove them for $D_1,..., D_{l-1}$, D'_l with $D'_l = D_l \cup \{k\}$, $\delta_1,..., \delta_l$, $\delta'_l = (\delta_l, \sigma_k)$, k being single site in $\partial\Omega \backslash \bigcup_{j=1}^{l} D_j$. We have (dropping the superscript Ω)

$$\begin{aligned}
&\frac{Z_{D_1 \cdots D'_l}(\delta_1,..., (\delta_l, \sigma_k), \tau)[Z(\tau)]^{l-1}}{[\prod_{j=1}^{l-1} Z_{D_j}(\delta_j, \tau)] \, Z_{D'_l}((\delta_l, \sigma_k), \tau)} \\
&\quad = \frac{Z_{D_1 \cdots D'_l}(\delta_1,..., (\delta_l, \sigma_k), \tau)[Z_{D'_l}((\tau_{D_l}, \sigma_k), \tau)]^{l-1}}{[\prod_{j=1}^{l-1} Z_{D_j, D'_l}(\delta_j, (\tau_{D_l}, \sigma_k), \tau)] \, Z_{D'_l}((\delta_l, \sigma_k), \tau)} \\
&\qquad \times \prod_{j=1}^{l-1} \left(\frac{Z_{D_j \cdot D'_l}(\delta_j, (\tau_{D_l}, \sigma_k), \tau) \, Z(\tau)}{Z_{D_j}(\delta_j, \tau) \, Z_{D'_l}((\tau_{D_l}, \sigma_k), \tau^{(k)})} \right) \\
&\quad = \frac{(Z_{D_1,..., D_l}(\delta_1,..., \delta_l, \tau^{(k)}))[Z(\tau^{(k)})]^{l-1}}{\prod_{j=1}^{l} Z_{D_j}(\delta_j, \tau^{(k)})} \prod_{j=1}^{l-1} \frac{Z_{D_j, k}(\delta_j, \sigma_k, \tau) \, Z(\tau)}{Z_{D_j}(\delta_j, \tau) \, Z_k(\sigma_k, \tau)} \\
&\quad \leqslant [1 + \mu(\rho)]^{\gamma(D_1,..., D_l)} [1 + \mu(\rho)]^{\sum_{j=1}^{l-1} |D_j|} \\
&\quad \left(\geqslant [1 - \mu(\rho)]^{\gamma(D_1,..., D_l)} [1 - \mu(\rho)]^{\sum_{j=1}^{l-1} |D_j|} \right)
\end{aligned} \tag{3.5}$$

where $\tau_x^{(k)} = \tau_x$ for $x \in \partial\Omega \backslash k$ and $\tau_k^{(k)} = \sigma_k$, $\rho = \min_{1 \leqslant i < j \leqslant l} \operatorname{dist}(D'_i, D'_j)$, $D'_i = D_i$, $i = 1,..., l-1$, $D'_l = D_l \cup \{k\}$. Since

$$\gamma(D_1,..., D_l) + \sum_{j=1}^{l-1} |D_j| = \gamma(D_l,..., D'_l)$$

the proposition is proven in this case.

(ii) We suppose that the estimates (3.2) are true for $D_1,..., D_l$, $\delta_1,..., \delta_l$ and we want to prove them for $D_1,..., D_l$, $D_{l+1} = \{k\}$, $\delta_1,..., \delta_l$, σ_k, k being a single site in $\partial\Omega \backslash \bigcup_{j=1}^{1} D_i$. We have

$$\frac{Z_{D_1,\ldots,D_l}(\delta_1,\ldots,\delta_l,\sigma_k,\tau)[Z(\tau)]^l}{[\prod_{j=1}^l Z_{D_j}(\delta_j,\tau)]\,Z_k(\sigma_k,\tau)}$$
$$=\frac{Z_{D_1,\ldots,D_l,k}(\delta_1,\ldots,\delta_l,\sigma_k,\tau)\{[Z(\sigma_k,\tau)]\}^{l-1}}{\prod_{j=1}^{l-1} Z_{D_{j,k}}(\delta_j,\sigma_k,\tau)}\prod_{j=1}^{l}\frac{Z_{D_{j,k}}(\delta_j,\sigma_k,\tau)\,Z(\tau)}{Z_{D_j}(\delta_j,\tau)\,Z_k(\sigma_k,\tau)}$$
$$=\frac{Z_{D_1,\ldots,D_l}(\delta_1,\ldots,\delta_l,\tau^{(k)})\{[Z(\tau^{(k)})]\}^{l-1}}{\prod_{j=1}^{l} Z_{D_j}(\delta_j,\tau^{(k)})}\prod_{j=1}^{l}\frac{Z_{D_{j,k}}(\delta_j,\sigma_k,\tau)\,Z(\tau)}{Z_{D_j}(\delta_j,\tau)\,Z_k(\sigma_k,\tau)}$$
$$\leqslant[1+\mu(\rho)]^{\gamma(D_1,\ldots,D_l)}[1+\mu(\rho))]^{\sum_{j=1}^l|D_j|}$$
$$(\geqslant[1-\mu(\rho)]^{\gamma(D_1,\ldots,D_l)}[1-\mu(\rho)]^{\sum_{j=1}^l|D_j|}) \tag{3.6}$$

but

$$\gamma(D_1,\ldots,D_l)+\sum_{j=1}^{l}|D_j|=\gamma(D_1,\ldots,D_l,D_k)$$

and so we conclude the proof. ▮

Corollary 3.2. Suppose that for the system described by the Hamiltonian given by Eq. (2.1) Condition A holds with

$$\lim_{x\to\infty}\mu(x)\,x^2=0 \tag{3.7}$$

Then for L sufficiently large, the bounds (2.41), (2.42) are satisfied for the corresponding polymer model and so the result of Theorem 2.2 applies to our system.

Proof. From definitions (2.17), (2.28), from Condition A, Eqs. (3.7), (2.7), and Proposition 3.1, we deduce that the inequalities (2.41) are satisfied with

$$\lambda=\lambda(L)=C_1L^2\mu(C_2L) \tag{3.8}$$

for some positive constants C_1, C_2. Since by Eq. (3.7), $\lambda(L)\to_{L\to\infty}0$, we get the result. ▮

Now we want to discuss the relationship between condition (3.1) and some conditions that have been used by Dobrushin and Shlosman to get uniqueness for lattice spin systems in the framework of their theory, which does not make use of the cluster expansion. Given a finite $\Omega\subset Z^2$ and two single sites $k, k'\in\partial\Omega$, we can write

$$\frac{Z^{\Omega}_{k,k'}(\sigma_k,\sigma_{k'},\tau)\,Z^{\Omega}(\tau)}{Z^{\Omega}_k(\sigma_k,\tau)\,Z^{\Omega}_{k'}(\sigma_{k'},\tau)}=\frac{\mu^{\tau}_{\Omega}(f^{(k)}f^{(k')})}{\mu^{\tau}_{\Omega}(f^{(k)})\,\mu^{\tau}_{\Omega}(f^{(k')})} \tag{3.9}$$

where μ_{Ω}^{τ} is the Gibbs measure in Ω with boundary condition τ [see Eq. (2.5)] and if $\Delta^{(k)}$ $\Delta^{(k')}$ are defined as

$$\begin{aligned} \Delta^{(k)} &= \{x \in \Omega : \operatorname{dist}(x, k) \leqslant r_0\} \\ \Delta^{(k')} &= \{x \in \Omega : \operatorname{dist}(x, k') \leqslant r_0\} \end{aligned} \tag{3.10}$$

then

$$f^{(k)}: S_{\Delta^{(k)}} \to R, \qquad f^{(k')}: S_{\Delta^{(k')}} \to R$$

are given by

$$\begin{aligned} f^{(k)}(\sigma_{\Delta^{(k)}}) &= \exp[W_{\Delta^{(k)},k}(\sigma_{\Delta^{(k)}}, \sigma_k) - W_{\Delta^{(k)},k}(\sigma_{\Delta^{(k)}}, \tau_k)] \\ f^{(k')}(\sigma_{\Delta^{(k')}}) &= \exp[W_{\Delta^{(k')},k}(\sigma_{\Delta^{(k')}}, \sigma_{k'}) - W_{\Delta^{(k')},k}(\sigma_{\Delta^{(k')}}, \tau_{k'})] \end{aligned} \tag{3.11}$$

Let $\Omega = \Omega_1 \cup \Omega_2$ ($\Omega_1 \cap \Omega_2 = \emptyset$) and let $\mu_{\Omega,\Omega_1}^{\tau}(\sigma_{\Omega_1} \mid \sigma_{\Omega_2})$ denote the Gibbs conditional measure on S_{Ω_1} given σ_{Ω_2} in Ω_2 (and τ in $\partial\Omega$). Given Ω_3 strictly contained in Ω_1, let $q_{\Omega,\Omega_3}^{\tau}(\sigma_{\Omega_3 \mid \Omega_2})$ denote the relativization of the measure $\mu_{\Omega,\Omega_1}^{\tau}$ to S_{Ω_3}, namely

$$q_{\Omega,\Omega_3}^{\tau}(\sigma_{\Omega_3} \mid \sigma_{\Omega_2}) = \sum_{\sigma_{\Omega_1 \setminus \Omega_3} \in S_{\Omega_1 \setminus \Omega_3}} \mu_{\Omega,\Omega_1}^{\tau}(\sigma_{\Omega_1 \setminus \Omega_3}, \sigma_{\Omega_3} \mid \sigma_{\Omega_2}) \tag{3.12}$$

Similarly, for Ω^* strictly contained in Ω

$$\mu_{\Omega,\Omega^*}^{\tau}(\sigma_{\Omega^*}) = \sum_{\Omega \setminus \Omega^*} \mu_{\Omega}^{\tau}(\sigma_{\Omega \setminus \Omega^*}, \sigma_{\Omega^*}) \tag{3.13}$$

We can write

$$\begin{aligned} \mu_{\Omega}^{\tau}(f^{(k)} f^{(k')}) &= \sum_{\sigma_\Omega} \mu_{\Omega,\Omega\setminus\Delta^{(k')}}^{\tau}(\sigma_{\Omega\setminus\Delta^{(k')}} \mid \sigma_{\Delta^{(k')}}) \\ &\quad \times \mu_{\Omega,\Delta^{(k')}}^{\tau}(\sigma_{\Delta^{(k')}}) f^{(k)}(\sigma_{\Delta^{(k)}}) f^{(k')}(\sigma_{\Delta^{(k')}}) \\ &= \sum_{\sigma_{\Delta^{(k)}} \sigma_{\Delta^{(k')}}} [q_{\Omega,\Delta^{(k)}}^{\tau}(\sigma_{\Delta^{(k)}} \mid \sigma_{\Delta^{(k')}}) \\ &\quad - \sum_{\sigma^*_{\Delta^{(k')}}} q_{\Omega,\Delta^{(k)}}^{\tau}(\sigma_{\Delta^{(k)}} \mid \sigma^*_{\Delta^{(k')}}) \mu_{\Omega,\Delta^{(k')}}^{\tau}(\sigma^*_{\Delta^{(k')}}) \\ &\quad + \sum_{\sigma^*_{\Delta^{(k')}}} q_{\Omega,\Delta^{(k')}}^{\tau}(\sigma_{\Delta^{(k)}} \mid \sigma^*_{\Delta^{(k')}}) \mu_{\Omega,\Delta^{(k)}}^{\tau}(\sigma^*_{\Delta^{(k')}})] \\ &\quad \times \mu_{\Omega,\Delta^{(k')}}^{\tau}(\sigma_{\Delta^{(k')}}) f^{(k)}(\sigma_{\Delta^{(k)}}) f^{(k')}(\sigma_{\Delta^{(k')}}) \\ &= \mu_{\Omega}^{\tau}(f^{(k)}) \mu_{\Omega}^{\tau}(f^{(k')}) \\ &\quad + \sum_{\sigma_{\Delta^{(k)}} \sigma_{\Delta^{(k')}} \sigma^*_{\Delta^{(k')}}} [q_{\Omega,\Delta^{(k)}}^{\tau}(\sigma_{\Delta^{(k)}} \mid \sigma_{\Delta^{(k')}}) - q_{\Omega,\Delta^{(k')}}^{\tau}(\sigma_{\Delta^{(k)}} \mid \sigma^*_{\Delta^{(k')}})] \\ &\quad \times \mu_{\Omega,\Delta^{(k')}}^{\tau}(\sigma^*_{\Delta^{(k')}}) \mu_{\Omega,\Delta^{(k')}}^{\tau}(\sigma_{\Delta^{(k')}}) f^{(k)}(\sigma_{\Delta^{(k)}}) f^{(k')}(\sigma_{\Delta^{(k')}}) \end{aligned} \tag{3.14}$$

Now suppose that the following condition B is satisfied.

Condition B. $\exists$ a decreasing function $\mu^*: R^+ \to R^+ : \forall$ finite $\Omega \subset Z^2$, $\forall k, k' \in \partial\Omega$:

$$\sup_{(\sigma_{\Delta(k)}\sigma_{\Delta(k')}\sigma^*_{\Delta(k')})} |q^{\tau}_{\Omega,\Delta^{(k)}}(\sigma_{\Delta^{(k)}} \mid \sigma_{\Delta^{(k')}}) - q^{\tau}_{\Omega,\Delta^{(k')}}(\sigma_{\Delta^{(k)}} \mid \sigma^*_{\Delta^{(k')}})| \leqslant \mu^*(\text{dist}(k, k')) \tag{3.15}$$

Then by Eqs. (3.9), (3.14) we see that condition A is satisfied with $\mu = M\mu^*$, where M is a suitable positive constant. On the other hand, it is easy to prove the inverse implication.

For given Ω, consider a set Δ in Ω: $\text{dist}(\Delta, \partial\Omega) = 1$ (namely Δ is adjacent from the interior to the boundary) and a site y in $\partial\Omega$.

Given $\tau \in S_{\partial(\Omega\setminus\Delta)}$, $\sigma_y, \sigma'_y \in \{0, 1,..., n\}$, consider the relativization to S_Δ of the Gibbs measure in Ω with boundary condition σ_y in y and τ in $\partial\Omega\setminus y$ (we simply write τ to denote the restrictions $\tau_{\partial\Omega}, \tau_{\partial\Omega\setminus y}$ as well):

$$q_{\Omega,\Delta}(\sigma_\Delta \mid \sigma_y \tau) = \sum_{\sigma_{\Omega\setminus\Delta} \in S_{\Omega\setminus\Delta}} \mu^{\tau}_{\Omega}(\sigma_{\Omega\setminus\Delta}, \sigma_\Delta)$$

If

$$f(\sigma_\Delta, \tau) = \exp[H_\Delta(\sigma_\Delta) + W_{\Delta,\partial\Omega}(\sigma_\Delta, \tau)]$$

and $Z^{\Omega\setminus\Delta}(\tau)$ is the partition function in $\Omega\setminus\Delta$ with boundary condition τ, we can write [if $\text{dist}(y, \Delta) > r_0$]

$$\begin{aligned} q_{\Omega,\Delta}(\sigma_\Delta \mid \sigma_y \tau) &= \frac{f(\sigma_\Delta, \tau)\, Z^{\Omega\setminus\Delta}_{\Delta,y}(\sigma_\Delta, \sigma_y, \tau)}{\sum_{\sigma'_\Delta} f(\sigma'_\Delta, \tau)\, Z^{\Omega\setminus\Delta}_{\Delta,y}(\sigma'_\Delta, \sigma_y, \tau)} \\ &= \left[f(\sigma_\Delta, \tau)\, Z^{\Omega\setminus\Delta}_{\Delta}(\sigma_y \tau) \frac{Z^{\Omega\setminus\Delta}_{\Delta,y}(\sigma_\Delta, \sigma_y, \tau)\, Z^{\Omega\setminus\Delta}(\tau)}{Z^{\Omega\setminus\Delta}_{\Delta}(\sigma_\Delta, \tau)\, Z^{\Omega\setminus\Delta}_{y}(\sigma_y, \tau)} \right] \\ &\quad \times \sum_{\sigma'_\Delta} f(\sigma'_\Delta, \tau)\, Z^{\Omega\setminus\Delta}_{\Delta}(\sigma_y \tau) \frac{Z^{\Omega\setminus\Delta}_{\Delta,y}(\sigma'_\Delta, \sigma_y, \tau)\, Z^{\Omega\setminus\Delta}(\tau)}{Z^{\Omega\setminus\Delta}_{\Delta}(\sigma'_\Delta, \tau)\, Z^{\Omega\setminus\Delta}_{y}(\sigma_y, \tau)} \Bigg]^{-1} \end{aligned} \tag{3.16}$$

If condition A is satisfied with μ: $\mu(1)$ is sufficiently small, then from Eq. (3.16) and Proposition 3.1, we deduce, $\forall \sigma_\Delta, \sigma_y, \sigma'_y, \tau$:

$$|q_{\Omega,\Delta}(\sigma_\Delta \mid \sigma_y \tau) - q_{\Omega,\Delta}(\sigma_\Delta \mid \sigma'_y \tau)| \leqslant 4 \frac{v^*(\text{dist}(\Delta, y))}{1 - v^*(\text{dist}(\Delta, y))^2} \tag{3.17}$$

where

$$v^* = \max[(1 + \bar{M}\mu)^{|\Delta|} - 1,\ 1 - (1 - \bar{M}\mu)^{|\Delta|}] \cdot \bar{M}$$

for a suitable positive constant $\bar{M}$, and so Condition B is satisfied with $\mu^* = \tilde{M}\nu^*$ ($\tilde{M}$ is a suitable constant). Condition B is implied, with $\mu^*(\rho) = \theta \cdot 4r_0^2 \exp(-\gamma\rho)$, by condition IIIc of Ref. 5:

Condition IIIc of Ref. 5. $\forall$ finite $\Omega \subset Z^2$, $\forall \Delta \subset \Omega$, $y \in \partial\Omega$, $\tau \in S_{\partial\Omega \setminus y}$

$$\sup_{\sigma_\Delta, \sigma_y, \sigma'_y} |q_{\Omega,\Delta}(\sigma_\Delta \mid \sigma_y \tau) - q_{\Omega,\Delta}(\sigma_\Delta \mid \sigma'_y \tau)| \leqslant \theta \exp[-\gamma \operatorname{dist}(\Delta, y)] \tag{3.18}$$

[see Eq. (2.24) of Ref. 5]. In Ref. 5 the authors prove analyticity properties on the basis of ten equivalent conditions, one of which is Condition IIIc. In particular, they give a so-called "constructive condition" (IIIe) in which the bound (3.18) is required to hold only for $|\Omega| \leqslant N$, for some N that can be explicitly estimated. The authors remark that this explicit estimate involves very complicated calculations. In our case, if we assume Condition B, with μ^* like μ in Eq. (3.7) *only* for $|\Omega| \leqslant N$, it is very easy to estimate the minimal N for which we get analyticity: it is sufficient to use Proposition 3.1 for $|\Omega| \leqslant N$ and then for λ as in Eq. (3.8) the inequality (2.41).

To conclude, we can say that our finite-size condition [Eqs. (2.40), (2.42)] can, at least in principle, be used to get a computer-assisted proof of analyticity and it is reasonable that for sufficiently large size it works in the intermediate temperature region.

On the other hand, it seems to be possible to prove theoretically for general ferromagnetic spin systems over T_c some weaker form of condition A and to improve our methods in order to get a general theoretical result.

NOTE ADDED

After the completion of this work I became aware of an interesting new paper by Dobrushin and Shlosman[12] in which they give explicit estimates of the minimal sizes for which their constructive condition has to be satisfied in order to get analyticity.

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